RAN-2203080301010002

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M.Sc. (Sem.-I) Examination November - 2025

Mathematics : Advanced Real Analysis (PGMTH-101)

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) All questions are compulsory. (3) Figures to the right indicate marks of the corresponding question.	Student's Signature

Q.1. Answer the following (any two) **14**

1. Prove that:
   (i) Finite intersection of open sets is open.
   (ii) Arbitrary union of open sets is open.
   Also, show that arbitrary intersection of open sets need not to be open.

2. Define an algebra of sets. Given any collection ***C*** of subsets of ***X***, prove that there exists a smallest algebra ***A*** containing ***C***.

3. Prove that: The outer measure of any interval is its length.

Q.2. Answer the following (any two) **14**

1. Let < E*n* > be an infinite decreasing sequence of measurable sets, that is $E_{n+1} \subset E_n$, for each *n* and mE_1 be finite. Prove that $m(\cap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} mEn$.

2. Prove that the set E is measurable if and only if given ε > 0, there is an open set $O \supset E$ with $m^*(O \sim E) < \varepsilon$.

3. Define the characteristic function X_A of A. Show that the function X_A is measurable if and only if A is measurable.

Q.3. Answer the following (any two) **14**

1. Let A be any set and $E_1, E_2, E_3, \ldots, E_n$ be an finite sequence of disjoint measurable sets then show that $m^*(A \cap [\cup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A \cap Ei)$.

2. Define sum modulo 1 (+) on [0,1). Let $E \subset [0,1)$ be measurable set, then prove that for any $y \in [0,1)$, the set $E + y$ is measurable and $m(E + y) = mE$.

3. Let f be an extended real valued function whose domain is measurable then show that the following statements are equivalent for each real number α :
   $\{x/f(x) > \alpha\}$ is measurable.
   $\{x/f(x) \geq \alpha\}$ is measurable.
   $\{x/f(x) < \alpha\}$ is measurable.
   $\{x/f(x) \leq \alpha\}$ is measurable.

Q.4. Answer the following (any two) **14**

1. Define Riemann Integral and Reimann integrable functions. Show that if

$$f(x) = \begin{cases} 0 & if\ x is rational \\ 1' & if\ x is irrational \end{cases} ;for\ x \in [a,b],$$

   then f is not Reimann integrable over $[a, b]$.

2. Define a simple function and describe the canonical representation of a simple function. Let $\varphi = \sum_{i=1}^{n} a_i X_{E_i}$, with $E_i \cap E_j = \emptyset$ for $i \neq j$. If each Ei is a measurable set of finite measure then show that
   $\int \varphi = \sum_{i=1}^{n} a_i\, m(E_i)$.

3. Let $\{f_n\}$ be a sequence of measurable functions defined on set E of finite measure and suppose that $| f_n (x) \leq M |$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E then prove that $\int_E f = \lim \int_E f_n$.

Q.5. Answer the following (any two) **14**

1. State and Prove Fatou's Lemma.

2. Let $\langle g_n \rangle$ be a sequence of integrable functions that converges almost everywhere to an integrable function g. let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and (f_n) converges to f almost everywhere. If $\int g = \lim \int g_n$ then prove that $\int f = \lim \int f_n$.

3. If $\langle f_n \rangle$ be an increasing sequence of non-negative measurable functions defined on a measurable set E such that $f_n \to f$ almost everywhere, then prove that $\int f = \lim \int f_n$.
